

# Breaking $so(4)$ symmetry without degeneracy lift

A. Pallares-Rivera, F. de J. Rosales-Aldape, M. Kirchbach

Instituto de Física  
 Universidad Autonoma de San Luis Potosí  
 Av. Manuel Nava 6, Zona Universitaria  
 SLP 78290 San Luis Potosí, México<sup>1</sup>

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**Abstract:** On the example of the quantum motion on  $S^3$  perturbed by the trigonometric Scarf potential (Scarf I) with one quantized and one continuous parameter we argue that the breakdown by external scales of a Lie-algebraic symmetry of a quantum system must not necessarily amount to a lift of the degeneracies in the spectrum. To be specific, we show that though the spectrum under discussion carries hydrogen-like degeneracies, the corresponding wave functions do not transform according to  $so(4)$  carrier spaces but are finite linear combinations of  $so(4)$  representation functions describing carrier spaces of different dimensionality. Alternatively, these decompositions are also expansions in powers of the symmetry breaking scale, and allow to quantitatively describe the order to which the symmetry is violated. We conclude on the general possibility to break perturbatively an intact Lie algebraic symmetry,  $so(4)$  in our case, by external scales, such as the continuous parameter of Scarf I, and without leaving trace in the spectra. In effect, degeneracy can throughout be compatible with symmetry breaking.

## 1 Symmetry and degeneracy: Introductory remarks

Symmetry and degeneracy are two concepts which one traditionally associates with the basics of the quantum mechanics teachings. One may think of the degeneracy with respect to the magnetic quantum number,  $m$ , of a quantum level of a given angular momentum  $\ell$  which is  $(2\ell + 1)$ -fold, and typical for the states bound within all central potentials. A more advanced example would be the degeneracy of the states within the levels describing the quantum motion within the Coulomb potential of an electron without spin, which is  $N^2$ -fold with  $N$  standing for the principal quantum number of the Coulomb potential problem. In the first case, the degeneracy is due to the rotational

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<sup>1</sup> e-mail: pallares@ifisica.uaslp.mx  
 e-mail: r\_felipedejesus@yahoo.com.mx  
 e-mail: mariana@ifisica.uaslp.mx

invariance of the three-dimensional position space, which requires conservation of angular momentum, and demands the total wave functions of the central potentials to be simultaneously eigenfunctions of  $\mathbf{L}^2$ , and  $L_z$ , with  $\mathbf{L}$  standing for the angular momentum pseudo-vector, and  $L_z$  for its  $z$ -component. As long as  $\mathbf{L}^2$  acts as the Casimir invariant of the  $so(3)$  algebra, the degeneracy with respect to the magnetic quantum number (the  $L_z$  eigenvalues  $m \in [-\ell, +\ell]$ ) is attributed to the rotational invariance of the Hamiltonian. The second case is bit more involved in so far as in order to explain the larger  $N^2$ -fold degeneracy, one needs to invoke the higher  $so(4)$  symmetry algebra underlying the Coulomb potential problem by accounting for the constancy of the Runge-Lenz vector next to that of angular momentum [1].

The list can be continued by some more examples, a popular one being the case of the Pöschl-Teller potential,  $V_{PT} = -\frac{\lambda(\lambda+1)}{\cosh^2 \eta}$ . In [2] it has been noticed that the free quantum motion on the one-sheeted two-dimensional hyperboloid,  $x^2 + y^2 - z^2 = 1^2$ , an  $AdS_2$  space [3], transforms, upon an appropriate change of variables, into the one-dimensional (1D) Schrödinger equation with same potential. As long as the kinetic-energy Laplace-Beltrami operator on the hyperboloid is proportional to the Casimir invariant of the  $so(2,1)$  isometry algebra of the  $AdS_2$  surface, the free motion on the curved space is described by means of the eigenvalue problem of that very Casimir invariant and exhibits the corresponding characteristic spectral patterns. In consequence, also the spectrum of the 1D-Schrödinger equation with the Pöschl-Teller potential is classified according to the irreducible representations of the same algebra and carries patterns identical to those of the free motion on  $AdS_2$ .

The idea, that the spectrum of a Schrödinger equation with a given potential exhibits certain Lie algebraic degeneracies because in some appropriately chosen variables it becomes identical to the eigenvalue problem of a Casimir invariant of the algebra in question, has been further elaborated and generalized by many authors (see [4] for a review). It has become known in the literature under the name of “symmetry algebra of a potential”, or, simply, “potential algebra”. The potential algebra concept has been quite successful in relating degeneracy patterns of a potential to an underlying algebraic symmetry, always when such has been possible. Predominantly, the  $su(1,1)$  symmetry of the Natanzon-class potentials has been extensively studied within this context, for example in refs. [5], [6], [7], [8], among others, but also the  $so(4)$  symmetry of the trigonometric Pöschl-Teller potential [9], on the one side, and of the trigonometric Rosen-Morse potential [10], [11] on the other side, has been paid due attention. In effect, the observation of degeneracies in the spectrum of a given potential problem that appear patterned after a known symmetry algebra, as a rule awakes the expectation that same algebra may determine the symmetry of the potential in question.

The view on degeneracy as a consequence of symmetry is perhaps one of the most strained colloquial wisdom attributed to quantum mechanics. Yet, it is well known a fact that at the same time quantum mechanics successfully describes also a broad range

of phenomena of degeneracy without symmetry, as the violation of the non-crossing rule in the correlated electron system of Benzene described by means of the Hubbard Hamiltonian [12], or the detection of resonance degeneracies in a double well potential [13]. Such type of degeneracies are ordinarily termed to as accidental, better, fortuitous. An indispensable text on the aspects of degeneracy without symmetry is provided by [14] within the context of quantum chaotic motion. Recapitulating the literature, degeneracy so far has been either associated with a symmetry, or, with the complete lack of such.

We here draw attention to a third option, namely, to the possibility of starting with a free quantum motion obeying a particular fundamental Lie algebra symmetry and violating it by a perturbation without lifting the unperturbed degeneracy patterns. We will show that the perturbation of the free quantum motion on  $S^3$  by the two-parameter potential,

$$V_{S^3}(\chi) = b^2 \sec^2 \chi - b(2\ell + 1) \tan \chi \sec \chi, \quad (1)$$

is  $so(4)$  violating, though degeneracy conserving. Here,  $\ell$  takes non-negative integer values (referred to as “quantized”), while  $b$  is continuous.

The paper is structured as follows. In the next section we briefly review for the sake of self-sufficiency of the presentation, the trigonometric Scarf potential problem on  $S^3$  with the emphasize on the hydrogen-like degeneracies in the corresponding spectrum. In section 3 we show that the wave functions,  $\psi_{\kappa\ell m}(\chi, \theta, \varphi)$ , of Scarf I do not transform as bases of irreducible  $so(4)$  representations but are mixtures of  $so(4)$  representation functions transforming according to  $(K + 1)^2$ -dimensional carrier spaces of  $so(4)$ , with  $K \in [\ell, \kappa] \in \mathbf{N}$ . We identify the  $b$  parameter as the symmetry breaking scale, and conclude on the fortuitous character of the degeneracy patterns in the spectrum. Also there, we draw attention to the fact that the aforementioned decompositions can alternatively be viewed as expansions in the powers of  $b$ , which allows quantitatively to keep track of the order to which the symmetry is violated. The paper closes with a brief summary of the results.

## 2 The trigonometric Scarf potential and its degeneracy patterns

The (periodic) trigonometric Scarf potential is of frequent use in the description of be it diatomic-, and poly-atomic molecules in solid-state physics, on the one side, or in molecular and poly-molecular systems in chemical physics, on the other. In several quantum systems it simulates reasonably well the average effect exercised by the interatomic(inter-molecular) interactions on a single atom (molecule). The potential is

characterized by two parameters only, is exactly solvable, and easy to handle with by computational soft-wares, all advantages that make it interesting to both theoretical studies and applications. Specifically in the present study, we focus on the peculiarity that the spectrum of Scarf I in (1) exclusively depends on the  $\ell$  parameter alone, while the importance of the  $b$  parameter confines to the level of the wave functions. For non-negative integer  $\ell$ -values, the spectrum of Scarf I shows typical hydrogen-like degeneracy and one may expect the potential to have  $so(4)$  as a symmetry algebra. This is true only restrictively, namely, only if the second parameter,  $b$ , which is irrelevant to the spectrum, has been properly quantized too. This case has been studied in the literature in great detail and the understanding has been gained that the corresponding wave functions transform as  $so(4)$  representation functions [6]. However, for continuous  $b$  values this is of course not to be so, though the spectrum, in remaining unaffected, still keeps exhibiting those very same  $so(4)$  degeneracy patterns. Therefore, the case of Scarf I with one quantized and one continuous parameter provides an intriguing template for studying the rare phenomenon of observing Lie-algebraic degeneracies without an underlying fundamental Lie-algebra symmetry. To throw more light on this issue, is the goal of the present study. To be specific, in the following we elaborate a quantitative description of the aforementioned symmetry violation by expanding the wave functions in powers of the symmetry breaking scale.

## 2.1 The general 1D Schrödinger equation with the two-parameter Scarf I potential

The one-dimensional Schrödinger Hamiltonian,  $H_{\text{ScI}}(\chi)$ , with the trigonometric Scarf potential, here denoted by  $V_{\text{ScI}}(\chi)$ , and its exact solutions [6] are very well known and given (here in dimensionless units  $\hbar^2/2MR^2 = 1$ ) by

$$H_{\text{ScI}}(\chi) U(\chi) = \left[ -\frac{d^2}{d\chi^2} + V_{\text{ScI}}(\chi) \right] U(\chi) = \epsilon U(\chi), \quad (2)$$

$$V_{\text{ScI}}(\chi) = \frac{b^2 + a(a+1)}{\cos^2 \chi} - \frac{b(2a+1) \tan \chi}{\cos \chi}, \quad (3)$$

$$b^2 = \frac{2MR^2 B^2}{\hbar^2}, \quad a(a+1) = \frac{2MR^2}{\hbar^2} A(A-1), \quad (4)$$

$$U(\chi) = \mathbf{F}^{-1}(\chi) \cos^{a+1} \chi P_n^{a-b+\frac{1}{2}, a+b+\frac{1}{2}}(\sin \chi), \quad (5)$$

$$\mathbf{F}^{-1}(\chi) = \left( \frac{1 + \sin \chi}{1 - \sin \chi} \right)^{\frac{b}{2}}, \quad (6)$$

$$\epsilon = (a+n+1)^2, \quad \epsilon = \frac{2MER^2}{\hbar^2}. \quad (7)$$

The dimensionless angular variable  $\chi$  is represented as,  $\chi = \frac{r}{R}$ , where  $r$  is a distance,  $R$  is a suited matching length parameter,  $E$  is the bound state energy in MeV,  $P_n^{\alpha,\beta}(\sin \chi)$  are the Jacobi polynomials.

## 2.2 The $so(4)$ algebra of Scarf I with two quantized parameters

The expression for the energy,  $\epsilon$ , in (7) is such that for a non-negative integer  $a = \ell \in \mathbf{N}$ , the spectrum exhibits a hydrogen-like  $(\ell + n + 1)^2$ -fold degeneracy which, as it will be explained below, is characteristic for an algebraic  $so(4)$  symmetry. Instead, for half-integer  $a$ -values,  $a = m - 1/2$ ,  $m \in \mathbf{N}$ , the energy takes the form,  $\epsilon = t(t + 1) + 1/4$ , with  $t = m + n$ , and the spectrum at first glance exhibits a lower  $[2t + 1]$ -fold rotational type degeneracy, with  $t$  formally behaving as 3D angular momentum value, and  $m$  acting as the subordinate “magnetic” quantum number. However, under the map,  $t = \kappa/2$ , one finds,  $\epsilon = (\kappa + 1)^2/4$  and the degeneracy is again converted to one of the  $so(4)$  type, though  $\kappa$  would be allowed to take only even values. This type of  $so(4)$  degeneracy in the spectrum of the 1D-Schrödinger equation with the trigonometric Scarf potential, has been studied in detail in [6] where also the second parameter,  $b$ , has been quantized in a way similar to  $a$ .

The Scarf I Hamiltonian parametrized in this particular fashion has been cast in terms of  $so(4)$  generators, constructed by means of an appropriate  $so(4) = so(3) \oplus so(3)$  decomposition, and upon introducing the two magnetic quantum numbers,  $m$ , and  $m'$  (related to the quantization of the respective  $a$  and  $b$  parameters) as auxiliary phases into the wave function according to,  $U(\chi) \rightarrow \exp(im\alpha + im'\beta)U(\chi)$ . Then the generators of the two  $so(3)$  algebras underlying  $so(4)$  have been designed on the basis of the Schrödinger ladder operators,  $A^\pm = -\partial_\chi \pm W_{ScI}$ , exploiting their property to factorize  $H_{ScI}$ , for the one algebra, and the super-symmetric partner,  $\tilde{H}_{ScI}$ , for the other. Here,  $W_{ScI} = -(a - 1)\tan \chi + b \sec \chi$  stands for the super-potential of  $V_{ScI}$ . Finally, the first algebra of the ladder operators has been then closed to  $so(3)$  by  $(-i\partial_\alpha)$ , and the second by  $(-i\partial_\beta)$ . The property of the Schrödinger ladder operators to partake a closed Lie algebra under certain restrictions on the values of the potential parameters, provides any time that such is possible, a powerful method for the algebraic description of various quantum mechanical problems such as those related to the Coulomb-, the Harmonic-Oscillator and other interactions [7].

We here instead deal with a differently parametrized Hamiltonian, to which the approach of [6] does not apply. Namely, in the case of our interest, only the  $a$ -parameter is quantized, while the  $b$ -parameter has been left fully unrestricted.

### 3 Perturbing the free quantum motion on $S^3$ by Scarf I with one quantized parameter

From now onwards we assume the parameter  $a$  to be quantized, i.e. to take only discrete values according to  $a = \ell$  with  $\ell$  non-negative integer. The second parameter,  $b$ , will be kept continuous. Changing variable in (2)–(7) to,

$$\begin{aligned} U(\chi) &= \cos \chi \phi_{\kappa\ell}(\chi), & \phi_{\kappa\ell}(\chi) &= \mathbf{F}^{-1}(\chi) \cos^\ell \chi P_n^{\ell-b+\frac{1}{2}, \ell+b+\frac{1}{2}}(\sin \chi), \\ a &= \ell \in \mathbf{N}, \end{aligned} \quad (8)$$

and incorporating the whole 4-space, the equation (7) is straightforwardly transformed to a quantum motion on  $S^3$  as,

$$\begin{aligned} \left( -\Delta_{S^3} + 1 + \frac{b^2}{\cos^2 \chi} - \frac{b(2\ell+1)\tan \chi}{\cos \chi} \right) \phi_{\kappa\ell}(\chi) Y_\ell^m(\theta, \varphi) \\ = (\kappa+1)^2 \phi_{\kappa\ell}(\chi) Y_\ell^m(\theta, \varphi), \\ -\Delta_{S^3} = -\frac{1}{\cos^2 \chi} \frac{\partial}{\partial \chi} \cos^2 \chi \frac{\partial}{\partial \chi} + \frac{\mathbf{L}^2}{\cos^2 \chi}, \\ \mathbf{L}^2 = - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right], \quad \kappa = \ell + n \in \mathbf{N}, \quad \ell \in [0, \kappa]. \end{aligned} \quad (9)$$

Here,  $\Delta_{S^3}$  denotes the Laplace-Beltrami (kinetic energy) operator on  $S^3$ , whose expression corresponds to the following parametrization of the hypersphere in global coordinates,

$$\begin{aligned} \mathbf{r}^2 + x_4^2 &= R^2, & \mathbf{r}^2 &= x^2 + y^2 + z^2 \\ \mathbf{r} &= R \cos \chi, & x_4 &= R \sin \chi, \quad \chi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]. \end{aligned} \quad (10)$$

The matching length parameter,  $R$ , in (7) now takes the place of the constant radius of  $S^3$ ,  $\chi$  acquires meaning of a second polar angle, while  $r$  in (7) becomes the arc,  $\widehat{r}$ , measuring the distance from the equator to the North pole along the geodesic on the three-sphere,  $S^3$ . It is obvious that Schrödinger's 1D- Scarf I potential problem in eqs. (2)–(7) has been transformed into a quantum motion on  $S^3$ , perturbed by the  $b$ -dependent piece of the potential,

$$V_{S^3}(\chi) = \frac{b^2}{\cos^2 \chi} - \frac{b(2\ell+1)}{\cos \chi} \tan \chi. \quad (11)$$

The latter is the potential that we shall refer to from now onwards as Scarf I on  $S^3$ . The  $\ell(\ell+1)/\cos^2 \chi$  term, originally part of the flat-space  $V_{ScI}(\chi)$  in (7), has been absorbed

into  $\Delta_{S^3}$ , the kinetic energy operator on  $S^3$ . Apparently, for  $b = 0$  the free quantum motion on the curved surface is recovered. The equation (9) can also be viewed as the 4D quantum mechanical rigid rotator perturbed by  $V_{S^3}(\chi)$ , a problem of interest to diatomic– or dimolecular systems. In furthermore recalling the relationship between the Laplace-Beltrami operator and the operator of the squared 4D angular momentum,  $\mathcal{K}^2$ , a Casimir invariant of the  $so(4)$  isometry algebra of  $S^3$ , from here onwards for  $R = 1$ ,

$$-\Delta_{S^3} = \mathcal{K}^2, \quad (12)$$

allows to cast the free quantum motion on  $S^3$  in terms of the  $(\mathcal{K}^2 + 1)$  eigenvalue problem as

$$(\mathcal{K}^2 + 1) Y_{K\ell m}(\chi, \theta, \varphi) = (K + 1)^2 Y_{K\ell m}(\chi, \theta, \varphi), \quad (13)$$

with  $Y_{K\ell m}(\chi, \theta, \varphi)$  standing for the well-known 4D hyper-spherical harmonics, and with  $K \in [0, \infty)$ ,  $\ell \in [0, K]$ , and  $m \in [-\ell, +\ell]$ .

Here,  $\mathcal{K}^2$  is expressed in terms of the six generators  $J_i$  and  $A_i$  with  $i = 1, 2, 3$ , spanning the  $so(4)$  algebra [16],

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [A_i, A_j] = i\epsilon_{ijk} J_k, \quad [J_i, A_k] = i\epsilon_{ijk} A_k, \quad (14)$$

as

$$\mathcal{K}^2 = 2 (\mathbf{J}^2 + \mathbf{A}^2). \quad (15)$$

The canonical  $so(4)$  representation functions are the hyper-spherical harmonics,  $Y_{K\ell m}(\chi, \theta, \varphi)$ . They describe the  $(K + 1)^2$ -dimensional  $so(4)$  carrier spaces and are defined as,

$$Y_{K\ell m}(\chi, \theta, \varphi) = \mathcal{S}_{K\ell}(\chi) Y_\ell^m(\theta, \varphi), \quad \mathcal{S}_{K\ell}(\chi) = \cos^\ell \chi \mathcal{G}_{n=K-\ell}^{\ell+1}(\sin \chi), \quad (16)$$

where  $\mathcal{G}_{n=K-\ell}^{\ell+1}(\sin \chi)$  stand for the Gegenbauer polynomials. The  $\mathcal{S}_{K\ell}(\chi)$  functions are sometimes referred to as the “quasi-radial” functions of the free motion [15]. In now substituting  $(-\Delta_{S^3})$  in (9) by  $\mathcal{K}^2$ , the equation of the perturbed motion on  $S^3$  which we will be dealing with here, takes the following final shape,

$$\begin{aligned} \mathcal{H} \psi_{\kappa\ell m}(\chi, \theta, \varphi) &= \epsilon_K \psi_{\kappa\ell m}(\chi, \theta, \varphi), \\ \mathcal{H} &= \left( \mathcal{K}^2 + 1 + V_{S^3}(\chi) \right), \\ \mathcal{K}^2 &= -\frac{1}{\cos^2 \chi} \frac{\partial}{\partial \chi} \cos^2 \chi \frac{\partial}{\partial \chi} + \frac{\mathbf{L}^2}{\cos^2 \chi}, \\ \psi_{\kappa\ell m}(\chi, \theta, \varphi) &= \phi_{\kappa\ell}(\chi) Y_\ell^m(\theta, \varphi) \\ &\equiv \mathbf{F}^{-1}(\chi) \cos^\ell \chi P_{n=\kappa-\ell}^{\alpha, \beta}(\sin \chi) Y_\ell^m(\theta, \varphi), \\ \alpha &= \ell - b + \frac{1}{2}, \quad \beta = \ell + b + \frac{1}{2}, \end{aligned} \quad (17)$$

and with  $V_{S^3}(\chi)$  from (11). The energy excitations are,

$$\epsilon_\kappa = (\kappa + 1)^2, \quad \kappa = \ell + n \in \mathbf{N}, \quad \kappa \in [0, \infty). \quad (18)$$

The remarkable aspect of the trigonometric Scarf potential is that its spectrum does not depend at all on the perturbation parameter  $b$ , which can be as well infinitesimally small, as finite. This spectrum is characterized by a  $(\kappa + 1)^2$ -fold degeneracy of the states in a level, and formally copies the  $(\mathcal{K}^2 + 1)$ -eigenvalue problem.

The advantage of transforming eqs. (2)–(7) to a motion on  $S^3$  is that the free motion on the hypersphere is necessarily  $so(4)$  symmetric, and the symmetry breaking properties of the potential motion can then be easily studied by means of the well known perturbative techniques. Therefore, in depending on whether the perturbation retains or removes the  $so(4)$  symmetry of the kinetic motion, the spectrum of the potential problem under consideration will exhibit either stringent, or, fortuitous  $so(4)$  patterns. This contrasts the case of the radial Schrödinger equation in 3D flat space, where the symmetry of the free motion is only translational, while the perturbation eventually invokes  $so(4)$  in a more sophisticated manner [6].

### 3.1 Formulation of the problem and the goal

The formal coincidence of the spectrum of the trigonometric Scarf potential with that of the free quantum motion on  $S^3$  in (13) may awake the expectation that the Scarf Hamiltonian could have  $so(4)$  as a symmetry algebra. If this were to be so, then the corresponding wave functions would transform according to  $so(4)$  carrier spaces. Our point is that this is not to be the case.

We claim is that for a general  $\ell \neq \kappa$ , the wave functions  $\psi_{\kappa\ell m}(\chi, \theta, \varphi)$  of the perturbed motion on  $S^3$  from (17) do not behave as eigenfunctions of an  $so(4)$  algebra invariant. To prove this, we will show in the following that the wave functions  $\psi_{\kappa\ell m}(\chi, \theta, \varphi)$  in (17) are not related by a rotation to any  $so(4)$  representation functions spanning a given  $(\kappa + 1)^2$ -dimensional carrier space, but behave as mixtures of functions belonging to carrier spaces of different dimensionality.

For this it is sufficient to find one  $so(4)$  algebra realization such that  $\psi_{\kappa\ell m}(\chi, \theta, \varphi)$  decompose into functions representing carrier spaces of different dimensionality. Then, by virtue of the model independence of the Lie algebra, one is allowed to conclude that such a property can not be removed by any changes of the variables parametrizing the hypersphere. It is the goal of the subsequent section to construct such decompositions.



### 3.2 $so(4)$ symmetry violation in the Scarf I potential problem on $S^3$ without lifting the degeneracy of the free quantum motion

We first begin with calculating the representation functions of a realization of the  $so(4)$  algebra, namely the one spanned by the set of generators,

$$\begin{aligned}\tilde{J}_i &= \mathbf{F}^{-1} J_i \mathbf{F}, & \tilde{A}_i &= \mathbf{F}^{-1} A_i \mathbf{F}, \quad i = 1, 2, 3, \\ \tilde{\mathcal{K}}^2 &= \mathbf{F}^{-1} \mathcal{K}^2 \mathbf{F}, & \tilde{\mathcal{K}}^2 &= 2 \sum_{i=1}^3 \left( \tilde{J}_i^2 + \tilde{A}_i^2 \right),\end{aligned}\tag{19}$$

with  $\mathbf{F}^{-1}(\chi)$  from (6). The corresponding representation functions are “damped” hyper-spherical harmonics, defined as,

$$\begin{aligned}\tilde{Y}_{K\ell m}(\chi, \theta, \varphi) &= \mathbf{F}^{-1}(\chi) S_{K\ell}(\chi) Y_\ell^m(\theta, \varphi) = \tilde{S}_{K\ell}(\chi) Y_\ell^m(\theta, \varphi), \\ \tilde{S}_{K\ell}(\chi) &= \mathbf{F}^{-1}(\chi) S_{K\ell}(\chi).\end{aligned}\tag{20}$$

For the particular case of the parameter  $\ell$  taking its maximal value of  $\ell = \kappa$  (it includes the ground state  $\kappa = \ell = 0$ ), the polynomials in both the eqs. (8) and (20) are of zero degree, i.e. constants, and one encounters equality between the Scarf I solutions on  $S^3$ , and the representation functions of the  $so(4)$  algebra in (19), namely,

$$\psi_{\kappa, \ell=\kappa, m}(\chi, \theta, \varphi) = \tilde{Y}_{K=\kappa, \ell=\kappa, m}(\chi, \theta, \varphi) = \mathbf{F}^{-1}(\chi) Y_{K=\kappa, \ell=\kappa, m}(\chi, \theta, \varphi),\tag{21}$$

holds valid. For this particular case the solutions of the perturbed quantum motion on  $S^3$  under discussion result  $so(4)$  symmetric. However, for any other  $\ell \neq \kappa$  values, this is not to remain so. In the next section we examine the decomposition of the  $\psi_{\kappa\ell m}(\chi, \theta, \varphi)$  wave functions in (17) in the basis of  $\tilde{Y}_{K\ell m}(\chi, \theta, \varphi)$  in (20).

Our case is that for a general  $\ell \neq \kappa$ , the wave functions  $\psi_{\kappa\ell m}(\chi, \theta, \varphi)$  in (17) describing the perturbed motion on  $S^3$  are not eigenfunctions of the Casimir invariant,  $\tilde{\mathcal{K}}^2$ , of the  $so(4)$  algebra in the representation of the eq. (19), because they can not be related to the corresponding representation functions,  $\tilde{Y}_{K\ell m}(\chi, \theta, \varphi)$  in (20) by a rotation but behave as mixtures of the type  $\psi_{\kappa\ell m}(\chi, \theta, \varphi) = \sum_{K=\ell}^{K=\kappa} c_{K\ell}(b^\eta) \tilde{Y}_{K\ell m}(\chi, \theta, \varphi)$ , with  $\eta \in [0, (\kappa - \ell)]$ . Such a property is bound to remain representation independent by virtue of the model independence of the Lie group algebras.

### 3.3 The case $\ell = (\kappa - 1)$ as an illustrative example

Let us take a look on  $\ell = (\kappa - 1)$ , in which case the wave function of interest is given by

$$\phi_{\kappa(\kappa-1)}(\chi) = \mathbf{F}^{-1}(\chi) \cos^{(\kappa-1)} \chi P_1^{(\kappa-1)-b+\frac{1}{2}, (\kappa-1)+b+\frac{1}{2}}(\sin \chi).\tag{22}$$

Now the Jacobi polynomial allows for the following decomposition into Gegenbauer polynomials,

$$\begin{aligned} P_1^{(\kappa-1)-b+\frac{1}{2}, (\kappa-1)+b+\frac{1}{2}}(\sin \chi) &= -b + \frac{1}{2}(2\kappa+1)\sin \chi \\ &= -b \mathcal{G}_0^\kappa(\sin \chi) + \frac{(2\kappa+1)}{4\kappa} \mathcal{G}_1^\kappa(\sin \chi). \end{aligned} \quad (23)$$

In noticing that by the aid of eq. (16),

$$\cos^{\kappa-1} \chi \mathcal{G}_0^\kappa(\sin \chi) = S_{(\kappa-1)(\kappa-1)}(\chi), \quad \cos^{\kappa-1} \chi \mathcal{G}_1^\kappa(\sin \chi) = S_{\kappa(\kappa-1)}(\chi), \quad (24)$$

allows to equivalently rewrite eq. (22) as

$$\phi_{\kappa(\kappa-1)}(\chi) = -b \tilde{S}_{(\kappa-1)(\kappa-1)}(\chi) + \frac{(2\kappa+1)}{4\kappa} \tilde{S}_{\kappa(\kappa-1)}(\chi), \quad (25)$$

with  $\tilde{S}_{K\ell}(\chi)$  defined in (20).

In consequence, same relationship holds valid at the level of the total wave-, and representation functions,

$$\psi_{\kappa(\kappa-1)m}(\chi, \theta, \varphi) = -b \tilde{Y}_{(\kappa-1)(\kappa-1)m}(\chi, \theta, \varphi) + \frac{(2\kappa+1)}{4\kappa} \tilde{Y}_{\kappa(\kappa-1)m}(\chi, \theta, \varphi). \quad (26)$$

In effect, we observe that  $\psi_{\kappa(\kappa-1)m}(\chi, \theta, \varphi)$  contains a mixture of the lower dimensional carrier  $so(4)$  space,  $\tilde{Y}_{(\kappa-1)(\kappa-1)m}(\chi, \theta, \varphi)$ , i.e.,

$$\psi_{\kappa\ell m}(\chi, \theta, \varphi) = \sum_{K=\ell}^{K=\kappa} c_{K\ell}(b^{\kappa-K}) \tilde{Y}_{K\ell m}(\chi, \theta, \varphi), \quad \ell = \kappa - 1 \quad (27)$$

with the expansion coefficients  $c_{K\ell}(b^{\kappa-K})$  being uniquely fixed through the decomposition of the Jacobi into Gegenbauer polynomials. Therefore,  $so(4)$  can not be a symmetry algebra of the Scarf I potential on  $S^3$  for  $\ell = \kappa - 1$ . It is straightforward to verify that this statement is true for any  $\ell \neq \kappa$ , and to observe that the wave functions of a motion on  $S^3$ , perturbed by the trigonometric Scarf potential, always represent themselves as mixtures of  $so(4)$  representation functions corresponding to  $so(4)$  carrier spaces of different dimensionality. Examples are listed in the Table 1. For this reason, we conclude that the  $so(4)$  symmetry algebra of the free motion on  $S^3$  is violated by the perturbation due to the trigonometric Scarf potential, though the original degeneracy patterns are preserved by chance. The generalization of eq. (26) to any  $\ell$  reads

$$\psi_{\kappa\ell m}(\chi, \theta, \varphi) = \sum_{K=\ell}^{K=\kappa} c_{K\ell}(b^\eta) \tilde{Y}_{K\ell m}(\chi, \theta, \varphi), \quad \eta \in [0, (\kappa - \ell)]. \quad (28)$$

Notice summation over the  $K$ -index defining the dimensionality of the  $so(4)$  carrier spaces. For the lowest  $\ell$  values,  $\ell = \kappa, (\kappa - 1), (\kappa - 2)$ , one finds  $\eta = \kappa - K$ .

$\kappa$	$\ell$	$\phi_{\kappa\ell}(\chi)$	$=$	$\sum_{K=\ell}^{K=\kappa} c_{K\ell}(b^\eta) \tilde{S}_{K\ell}(\chi)$
$\kappa$	$\kappa$	$\phi_{\kappa\kappa}(\chi)$	$=$	$\tilde{S}_{\kappa\kappa}(\chi)$
$\kappa$	$(\kappa-1)$	$\phi_{\kappa(\kappa-1)}(\chi)$	$=$	$\frac{2\kappa+1}{4\kappa} \tilde{S}_{\kappa(\kappa-1)}(\chi) - b \tilde{S}_{(\kappa-1)(\kappa-1)}(\chi)$
$\kappa$	$(\kappa-2)$	$\phi_{\kappa(\kappa-2)}(\chi)$	$=$	$\frac{1}{8} \frac{(2\kappa+1)}{(\kappa-1)} \tilde{S}_{\kappa(\kappa-2)}(\chi) - \frac{b}{2} \frac{\kappa}{(\kappa-1)} \tilde{S}_{(\kappa-1)(\kappa-2)}(\chi) + \frac{b^2}{2} \tilde{S}_{(\kappa-2)(\kappa-2)}(\chi)$
$\kappa$	$(\kappa-3)$	$\phi_{\kappa(\kappa-3)}(\chi)$	$=$	$\frac{1}{32} \frac{4\kappa^2-1}{(\kappa-2)(\kappa-1)} \tilde{S}_{\kappa(\kappa-3)}(\chi) - \frac{b}{8} \frac{(2\kappa-1)\kappa}{(\kappa-1)(\kappa-2)} \tilde{S}_{(\kappa-1)(\kappa-3)}(\chi)$ $+ \frac{b^2}{8} \frac{(2\kappa-1)}{(\kappa-2)} \tilde{S}_{(\kappa-2)(\kappa-3)}(\chi) - \frac{b}{24} \left[ \frac{4b^2(\kappa-1)+(2\kappa+1)}{(\kappa-1)} \right] \tilde{S}_{(\kappa-3)(\kappa-3)}(\chi)$

Table 1: Decomposition of the “quasi-radial” wave functions,  $\phi_{\kappa\ell}(\chi)$ , of Scarf I in eq. (17) in the basis of the “quasi-radial” parts,  $\tilde{S}_{K\ell}(\chi)$  of the “damped” hyper spherical harmonics,  $\tilde{Y}_{K\ell m}(\chi, \theta, \varphi)$ , in (20). It is well visible that the Scarf I solutions represent themselves as mixtures of representation functions describing  $so(4)$  carrier spaces of different dimensionality, thus making the  $so(4)$  symmetry breaking manifest. The decompositions can alternatively be viewed as finite power series of the symmetry breaking scale  $b$ . Notice that the leading order ( $\mathcal{O}(b^0)$ ) terms respect the symmetry, as it should be. In this fashion, a quantitative scheme is elaborated which allows to keep track of the order to which the symmetry has been broken.

Finally, a comment is in order on the reason for which the perturbation of the quantum motion on  $S^3$  by Scarf I nonetheless happens to conserve the  $so(4)$  degeneracies. The issue is that the  $\chi$  dependent (quasi-radial) part of the equation (17) can be expressed in terms of  $\tilde{\mathcal{K}}^2$  as

$$\begin{aligned}
\left( \mathcal{K}^2 + 1 + V_{S^3}(\chi) \right) \phi_{\kappa\ell}(\chi) &= \left( \tilde{\mathcal{K}}^2 + 1 \right) \phi_{\kappa\ell}(\chi) - \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial P_{n=\kappa-\ell}^{\alpha,\beta}(\sin \chi)}{\partial \chi} \\
&= (\kappa+1)^2 \phi_{\kappa\ell}(\chi)
\end{aligned} \tag{29}$$

This equation is straightforwardly obtained from (17) by dragging the  $\mathbf{F}^{-1}(\chi)$  part of the wave function from the very right to the very left and making explicit use of the known solutions in the evaluation of all differentiations.

Though  $\phi_{\kappa\ell}(\chi)$  by themselves do not behave as  $\tilde{\mathcal{K}}^2$  eigenfunctions, the contributions of the gradient term are such that one ends up with the common  $(\kappa+1)^2$  eigenvalue and  $(\kappa+1)^2$ -fold degeneracies of the states in a level. Take for example the  $\ell = (\kappa-1)$  case already considered in (25) above. Substitution into (29) amounts to:

$$\begin{aligned}
\left(\mathcal{K}^2 + 1 + V_{S^3}(\chi)\right) \phi_{\kappa(\kappa-1)}(\chi) &= \left(\tilde{\mathcal{K}}^2 + 1\right) \phi_{\kappa(\kappa-1)}(\chi) - \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial P_1^{\alpha,\beta}(\sin \chi)}{\partial \chi} \\
&= ((\kappa - 1) + 1)^2 (-b) \tilde{S}_{(\kappa-1)(\kappa-1)}(\chi) \\
&\quad + (\kappa + 1)^2 \frac{(2\kappa + 1)}{4\kappa} \tilde{S}_{\kappa(\kappa-1)}(\chi) \\
&\quad + (2\kappa + 1)(-b) \tilde{S}_{(\kappa-1)(\kappa-1)}(\chi) \\
&= (\kappa + 1)^2 \phi_{\kappa(\kappa-1)}(\chi).
\end{aligned} \tag{30}$$

In the full space, obtained by multiplying (30) by  $Y_{\kappa-1}^m(\theta, \varphi)$  from the right, the latter equation amounts to,

$$\begin{aligned}
\left(\tilde{\mathcal{K}}^2 + 1\right) \psi_{\kappa(\kappa-1)m}(\chi, \theta, \varphi) &- \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^{\kappa-1} \chi \frac{\partial P_1^{\alpha,\beta}(\sin \chi)}{\partial \chi} Y_{\kappa-1}^m(\theta, \varphi) \\
&= (\kappa + 1)^2 \psi_{\kappa(\kappa-1)m}(\chi, \theta, \varphi),
\end{aligned} \tag{31}$$

as it should be. This simple exercise shows that the  $(\kappa + 1)^2$ -fold degeneracy in the spectrum of the trigonometric Scarf potential with the  $a$  parameter quantized to non-negative integer values is accidental and not due to  $so(4)$  symmetry.

## 4 Conclusions

The present study has been devoted to the explanation of the emerging  $so(4)$ -type of degeneracy patterns in the spectrum of the diatomic (dimolecular) trigonometric Scarf potential for the case in which the parameter  $a$  in (7) was allowed to take non-negative integer values, while  $b$  remained unrestricted. We showed that the motion on  $S^3$  perturbed by  $V_{S^3}(\chi)$  in (11) is  $so(4)$  symmetry violating, though degeneracy conserving. Our argumentation was based on the observation that the wave functions of the perturbed motion behaved as mixtures of functions, properly identified as genuine  $so(4)$  representation functions in (20), and which, as shown in (26), and the Table transformed as finite linear combinations of  $so(4)$  carrier spaces of different dimensionality. Alternatively, the decompositions present themselves as finite power series expansions in the  $b$  parameter, which permits one to quantitatively keep track of the order to which the symmetry is broken. Our findings are backed up by the established finite decompositions of the Jacobi polynomials, the key ingredients of the Scarf I solutions, in the basis of the Gegenbauer polynomials, the key ingredients of the canonical  $so(4)$  representation functions. It is because of these decompositions that the wave functions of the perturbed motion can not transform irreducibly under  $so(4)$ . The aim of the present investigation

has been to reveal possibility of retaining the degeneracy in the process of a symmetry violation through perturbation. The general interest in such an observation lies in the possibility to break fundamental Lie algebra symmetries by scales, such as temperatures, masses, lengths, without leaving trace in the spectra. Such a subtle symmetry breaking remains undetectable at the level of the energy excitations but it has inevitably to show up in the disintegration modes of the system.

In effect, the Lie algebraic transformation properties of the wave functions provide the more reliable criterion for a possible symmetry of a quantum mechanics problem than the degeneracy in the spectra.

In view of the wide use of the hyper-spherical geometry in the description of many-body systems such as Brownian motion [17], coherent states [18] etc. we expect the symmetry breaking mechanism reported here to acquire relevance.

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